Variable elimination, graph reduction and efficient g-formula

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Motivating example

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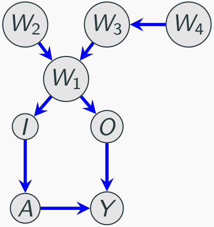

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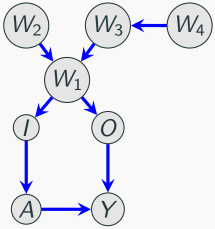

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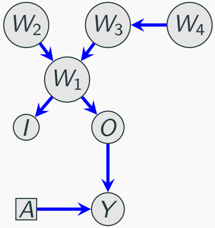

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☞ Due to the factorization of P, in the model, $\mathbb{E}\, Y(a)$ can be expressed **in different forms**.

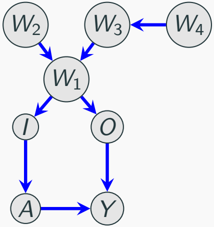
Other identifying formulae

We have back-door formulae

$$\Psi_a^{\mathrm{ADJ}}(P; \mathcal{G}) = \mathbb{E}[\mathbb{E}[Y \mid A = a, \mathbf{L}]],$$

where **adjustment set** $\mathbf{L}$ can take

$$\{O\}, \{I, O\}, \{I, W_1, O\}, \{I, O, W_1, W_2\}, \ldots.$$



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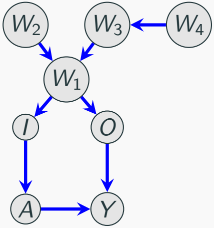

☞ Which formula should I use?

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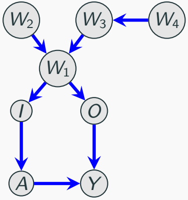

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W_2 \arrow[dr] & & W_3 \arrow[dl] & W_4 \arrow[l] \\
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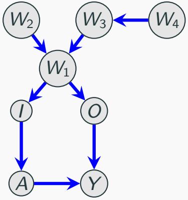

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☞ Yet, a simpler g-formula is **also** efficient!

$$\Psi_a(\mathbb{P}_n;\mathcal{G}^*) = \sum_{y,o,w_2,w_3} y\mathbb{P}_n[y \mid A=a,o]$$
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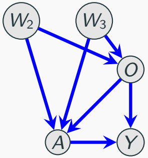
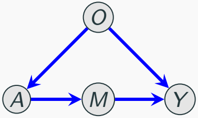


$\mathcal{G}^*$

Another example



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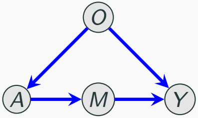

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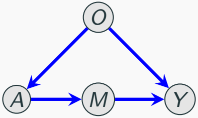

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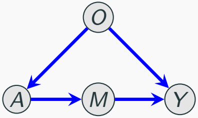

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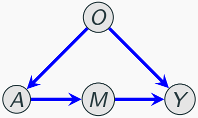

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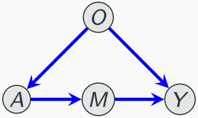

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☞ Which one should be preferred? *"Another area which is neglected in my opinion ... given an estimand find the best way of decomposing to estimate it"* — Judea Pearl (OCIS, Nov 17, 2020).

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☞ Observation: for some graphs, certain variables **"cancels out"** from $\chi_{\mathsf{eff},P}(\mathbf{V}; \mathcal{G})$.

Informative variables

Lemma (Rotnitzky and Smucler, 2020) Let $\mathcal{M}$ be a semiparametric model on vector $\mathbf{V}$. Suppose $\mathbf{V}'$ is a subvector of $\mathbf{V}$, such that

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Irreducible informative set $\mathbf{V}^*(\mathcal{G}) \equiv \{\text{smallest } \mathbf{V}' : \mathbf{V} \setminus \mathbf{V}' \text{ is uninformative}\}$.

Agenda

1. **Variable elimination**: identify informative variables $\mathbf{V}^*(\mathcal{G})$.
2. **Graph reduction**: characterize the marginal model over $\mathbf{V}^*(\mathcal{G})$.
3. Derive a **simpler g-formula**.

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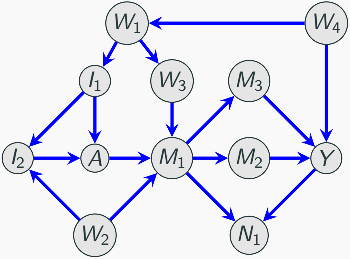

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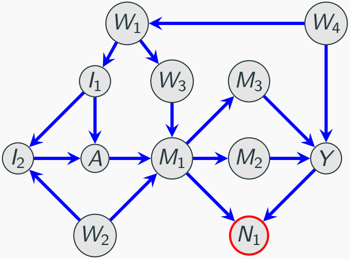

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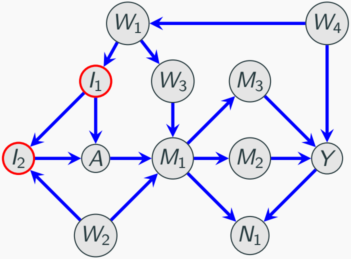

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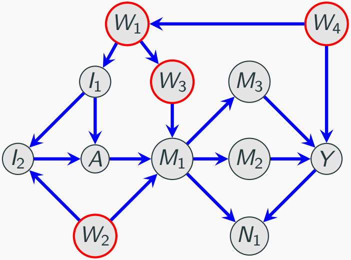

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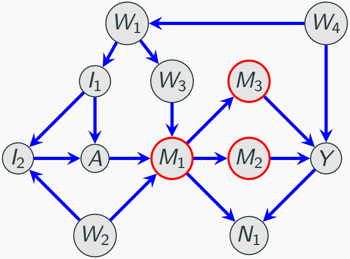

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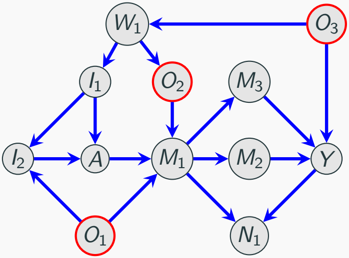

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☞ A special subset of **W** plays an important role:

$$\mathbf{O}(\mathcal{G}) := \mathrm{Pa}(\mathbf{M} \cup \{Y\}) \setminus (\mathrm{De}(\mathbf{M} \cup \{Y\}) \cup \{A\})$$

is the **optimal adjustment set** (Henckel, Perković, and Maathuis, 2022; Rotnitzky and Smucler, 2020).
(**O** consists of direct parents of **M** or Y that do not block causal paths.)

N and I are uninformative

Using conditional independences on $\mathcal{G}$, Rotnitzky and Smucler (2020) showed

$$\chi_{\text{eff},P}(\mathbf{V},\mathcal{G}) = \sum_{j=1}^{J} \left(\mathbb{E}[b(\mathbf{O}) \mid W_j, \text{Pa}(W_j)] - \mathbb{E}[b(\mathbf{O}) \mid \text{Pa}(W_j)]\right)$$
$$+ \sum_{k=1}^{K+1} \left(\mathbb{E}[AY/\pi(\mathbf{O}) \mid M_k, \text{Pa}(M_k)] - \mathbb{E}[AY/\pi(\mathbf{O}) \mid \text{Pa}(M_k)]\right),$$

where $M_{K+1} \equiv Y$, $b(\mathbf{O}) = \mathbb{E}[Y \mid A = 1, \mathbf{O}]$, $\pi(\mathbf{O}) = P(A = 1 \mid \mathbf{O})$.

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$$\chi_{\mathrm{eff},P}(\mathbf{V},\mathcal{G}) = \sum_{j=1}^{J}\left(\mathbb{E}[b(\mathbf{O}) \mid W_j, \mathrm{Pa}(W_j)] - \mathbb{E}[b(\mathbf{O}) \mid \mathrm{Pa}(W_j)]\right)$$
$$+ \sum_{k=1}^{K+1}\left(\mathbb{E}[AY/\pi(\mathbf{O}) \mid M_k, \mathrm{Pa}(M_k)] - \mathbb{E}[AY/\pi(\mathbf{O}) \mid \mathrm{Pa}(M_k)]\right),$$

where $M_{K+1} \equiv Y$, $b(\mathbf{O}) = \mathbb{E}[Y \mid A = 1, \mathbf{O}]$, $\pi(\mathbf{O}) = P(A = 1 \mid \mathbf{O})$.

Corollary This implies that

$$\mathbf{N}(\mathcal{G}) \equiv \{\text{non-ancestors of } Y\} \quad \text{and} \quad \mathbf{I}(\mathcal{G}) \equiv \{\text{indirect ancestors of } Y\}$$

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☞ Yet, $\mathbf{N}(\mathcal{G})$ and $\mathbf{I}(\mathcal{G})$ are defined with respect to $\mathcal{G}$ rather than $\mathcal{M}(\mathcal{G}, \mathbf{V})$.

Causal Markov equivalence

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Two DAGs $\mathcal{G}$ and $\mathcal{G}'$ are **Markov equivalent** if they define the same set of models

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Markov equivalence class (MEC) (Andersson, Madigan, and Perlman, 1997; Verma and Pearl, 1991)

$\mathcal{G} \simeq \mathcal{G}' \iff \mathcal{G}$ and $\mathcal{G}'$ share the same **adjacencies** and **unshielded colliders**

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☞ But Markov equivalence does not preserve the causal interpretation:

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Causal Markov equivalence (with respect to the effect of A on Y):

$$\mathcal{G} \overset{c}{\sim} \mathcal{G}' \iff \mathcal{G} \simeq \mathcal{G}' \text{ and } \Psi(P, \mathcal{G}) = \Psi(P, \mathcal{G}') \text{ for all } P \in \mathcal{M}_{\mathcal{G}}.$$

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☞ Vertices can be determined uninformative by moving **within** the causal Markov equivalence class (flipping edges).

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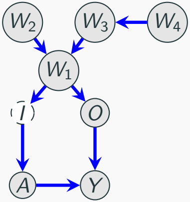

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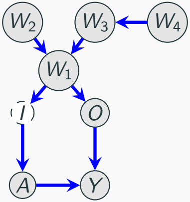
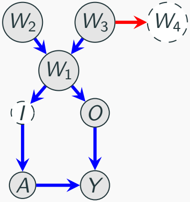
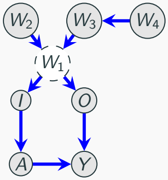


$\overset{c}{\sim}$



Not complete ...



$\mathcal{G}$

$\chi_{\text{eff},P}(\mathbf{V}, \mathcal{G})$ depends on W_1 thought the terms

$$\cancel{\mathbb{E}\{b_a(O) \mid W_1, W_2, W_3\}} \quad + \quad b_a(O) \quad - \quad \cancel{\mathbb{E}\{b_a(O) \mid W_1\}}$$

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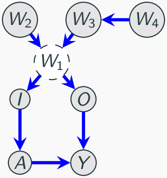

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☞ **Uninformative** W_1 cannot be detected this way!

Conspired cancellation

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Fix $W_j \equiv W_{j_0} \in \mathbf{W} \setminus \mathbf{O}$. Let $\mathbf{W} \cap \mathrm{Ch}(W_j)$ be topo-sorted as $\{W_{j_1}, \ldots, W_{j_r}\}$.

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☞ Then the EIF only depends on W_j through the terms:

$$+\mathbb{E}[b(\mathbf{O}) \mid W_j, \mathrm{Pa}(W_j)] + \mathbb{E}[b(\mathbf{O}) \mid W_{j_1}, \mathrm{Pa}(W_{j_1})] + \mathbb{E}[b(\mathbf{O}) \mid W_{j_2}, \mathrm{Pa}(W_{j_2})] \quad \cdots \quad + \mathbb{E}[b(\mathbf{O}) \mid W_{j_r}, \mathrm{Pa}(W_{j_r})]$$

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To *make this happen*, we can posit the following graphical **W-criterion**:

1. $W_j \perp\!\!\!\perp_{\mathcal{G}} \mathbf{O} \mid W_{j_r}, \mathrm{Pa}(W_{j_r}) \setminus \{W_j\}$,
2. and for $m = 1, \dots, r$:
   - (i) $W_{j_{m-1}} \to W_{j_m}$ (children are chained)
   - (ii) $\mathrm{Pa}(W_{j_m}) \subseteq \mathrm{Pa}(W_{j_{m-1}}) \cup \{W_{j_{m-1}}\}$ (parent sets are decreasing)
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☞ A similar graphical **M-criterion** applies to mediator $M_i \in \mathbf{M}$.

Complete graphical criteria

Theorem The set of informative variables is given by

$$\begin{aligned}\mathbf{V}^*(\mathcal{G}) = \{A, Y\} \cup \mathbf{O} \\ \cup \{W_j \in \mathbf{W} \setminus \mathbf{O} : W_j \text{ fails the W-criterion}\} \\ \cup \{M_i \in \mathbf{M} : M_i \text{ fails the M-criterion}\}.\end{aligned}$$

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3. W_j/M_i fails the W/M-criterion $\implies$ W_j/M_i is informative
   ☞ by constructing certain $P \in \mathcal{M}(\mathcal{G}, \mathbf{V})$ such that $\chi_{\text{eff},P}(\mathbf{V}, \mathcal{G})$ depends on W_j/M_i.

Graph reduction

☞ How do we represent the following **marginal model** of a DAG?

$$\mathcal{M}(\mathcal{G}, \mathbf{V}^*) \equiv \{P(\mathbf{V}^*) : P \in \mathcal{M}(\mathcal{G}, \mathbf{V})\},$$

where $\mathbf{V}^* \equiv \mathbf{V}^*(\mathcal{G})$.

☞ Marginal models of a DAG can be complicated.

Latent projection

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Suppose $\mathcal{G} = (\mathbf{V} \dot\cup \mathbf{U}, \mathbf{E})$ for observed $\mathbf{V}$ and latent $\mathbf{U}$.

1. Whenever there is a path of the form $w \rightarrow u_1 \rightarrow \cdots \rightarrow u_2 \rightarrow v$ add $w \rightarrow v$ (if not already present).
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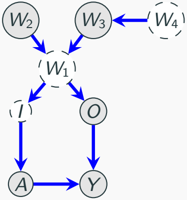

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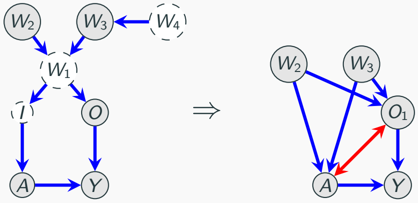

☞ Results in an ADMG (not always a DAG). Conceptually strange.

Graph reduction algorithm

$\mathbf{V}^* \leftarrow \{A, Y\} \cup \mathbf{W} \cup \mathbf{M}$

$\mathcal{G}^* \leftarrow \mathcal{G}(\mathbf{V}^*)$ by projecting out $\mathbf{N}$ and $\mathbf{I}$ with latent projection

for $v \in \mathbf{V}^*$ **do**

  if ($v \in \mathbf{W} \setminus \mathbf{O}$ and v satisfies the W-criterion) **or** ($v \in \mathbf{M}$ and v satisfies the M-criterion) **then**

    $\mathbf{V}^* \leftarrow \mathbf{V}^* \setminus \{v\}$

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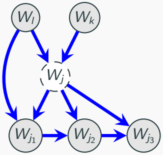

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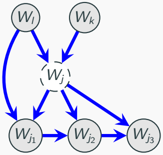

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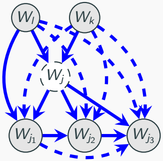


Graph reduction algorithm

$\mathbf{V}^* \leftarrow \{A, Y\} \cup \mathbf{W} \cup \mathbf{M}$

$\mathcal{G}^* \leftarrow \mathcal{G}(\mathbf{V}^*)$ by projecting out $\mathbf{N}$ and $\mathbf{I}$ with latent projection

for $v \in \mathbf{V}^*$ **do**

  if ($v \in \mathbf{W} \setminus \mathbf{O}$ and v satisfies the W-criterion) **or** ($v \in \mathbf{M}$ and v satisfies the M-criterion) **then**

    $\mathbf{V}^* \leftarrow \mathbf{V}^* \setminus \{v\}$

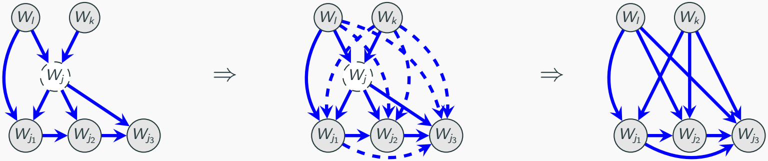
    $\mathcal{G}^* \leftarrow \mathcal{G}^*_{-v}$

  end if

end for

return $\mathcal{G}^*$

☞ $\mathcal{G}^*_{-v}$ saturates edges from $\mathrm{Pa}(v)$ to $\mathrm{Ch}(v)$ and those within $\mathrm{Ch}(v)$, before removing v.



Graph reduction algorithm

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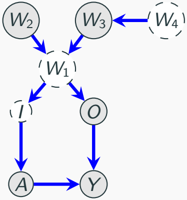
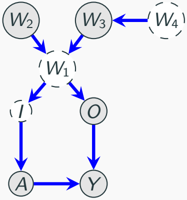


$\mathcal{G}$

$\Rightarrow$

Graph reduction algorithm



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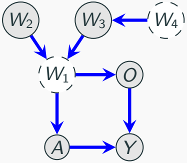

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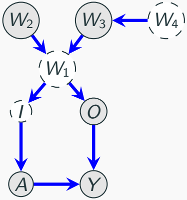

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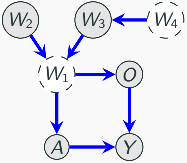

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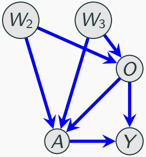
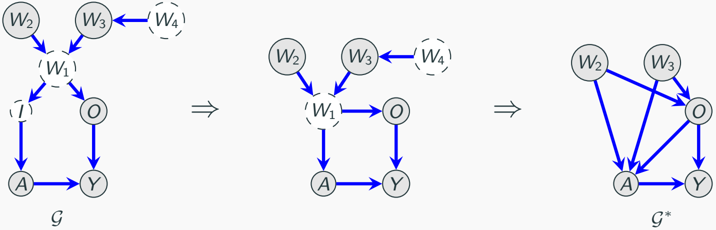


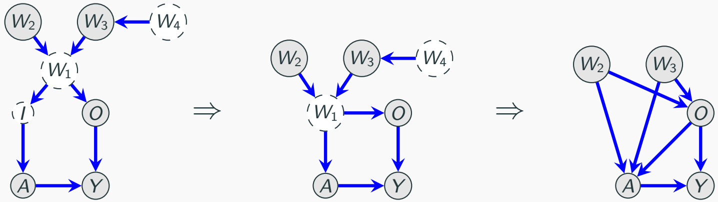
$\mathcal{G}^*$

Graph reduction algorithm



Theorem The reduced graph $\mathcal{G}^*$ is a DAG on vertices $\mathbf{V}^* \equiv \mathbf{V}^*(\mathcal{G})$ with the following properties.

Graph reduction algorithm



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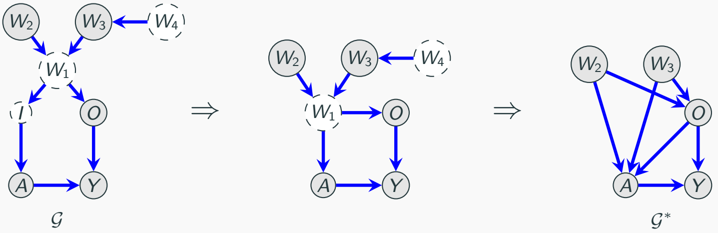

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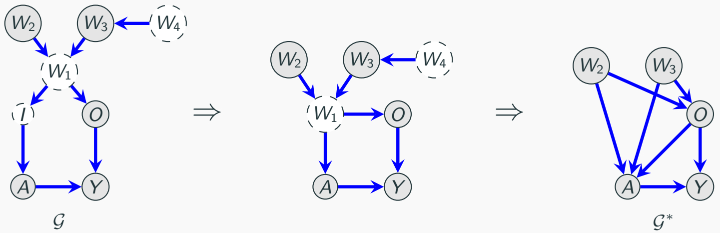
Graph reduction algorithm



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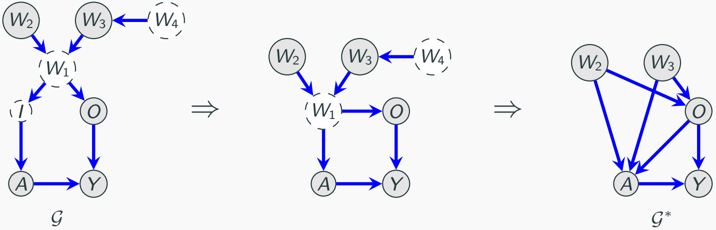
Graph reduction algorithm



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4. $\chi_{\text{eff},P}(\mathbf{V}, \mathcal{G}) = \chi_{\text{eff},P(\mathbf{V}^*)}(\mathbf{V}^*, \mathcal{G}^*)$ P-a.e. for every $P \in \mathcal{M}(\mathcal{G}, \mathbf{V})$.

Simplest efficient g-formula

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Corollary $\Psi_a(P; \mathcal{G}^*)$ is the irreducible "efficient" g-formula in the sense that

$$|\Psi_a(\mathbb{P}_n; \mathcal{G}^*) - \Psi_a(\mathbb{P}_n; \mathcal{G})| = o_p(n^{-1/2}) \quad \text{as } n \to \infty$$

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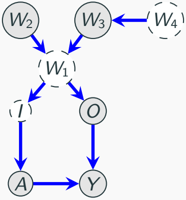

$\mathcal{G}$

$$\Psi_a(P;\mathcal{G}) = \sum_{y,o,i,w_1,w_2,w_3,w_4} yp(y \mid A=a, o)p(i \mid w_1)p(o \mid w_1) \times p(w_1 \mid w_2, w_3)p(w_3 \mid w_4)p(w_2)p(w_4).$$

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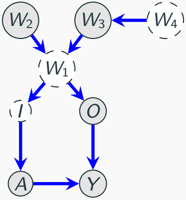

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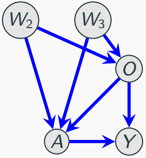
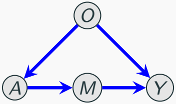


$\mathcal{G}^*$

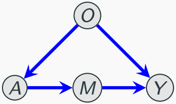
$$\Psi_a(P;\mathcal{G}^*) = \sum_{y,o,w_2,w_3} yP[y \mid A = a, o] \times P(o \mid w_2, w_3)P(w_2)P(w_3).$$

Examples (i)



$\mathcal{G} = \mathcal{G}^*$

Examples (i)



$\mathcal{G} = \mathcal{G}^*$

The g-formula

$$\Psi_a(P;\mathcal{G}) = \sum_{m,o} \mathbb{E}[Y \mid m, o] p(o) p(m \mid a) \text{ is efficient.}$$

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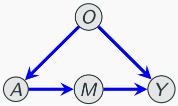

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☞ Neither the back-door

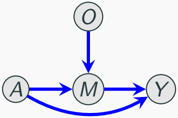
$$\Psi_a^{\text{ADJ}}(P;\mathcal{G}) = \sum_{o} \mathbb{E}[Y \mid a, o]p(o)$$

nor the front-door

$$\Psi_a^{\text{FRONT}}(P;\mathcal{G}) = \sum_{m} \left\{ \sum_{a} \mathbb{E}[Y \mid m, a']p(a') \right\} p(m \mid a)$$

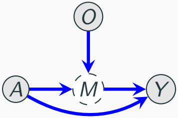
is efficient.

Examples (ii.a)



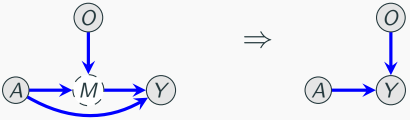
$\mathcal{G}_1$

Examples (ii.a)



$\mathcal{G}_1$

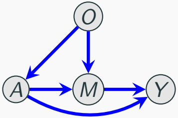
Examples (ii.a)



$\mathcal{G}_1$ $\Rightarrow$ $\mathcal{G}_1^*$

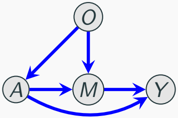
$$\Psi_a(P; \mathcal{G}_1^*) = \sum_o \mathbb{E}[Y \mid A = a, O = o] P(o).$$

Examples (ii.b)



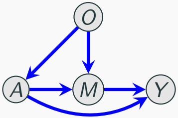
$\mathcal{G}_2$

Examples (ii.b)



$\mathcal{G}_2$

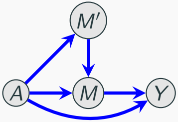
$\Rightarrow$



$\mathcal{G}_2^* = \mathcal{G}_2$

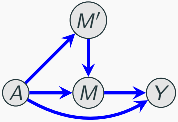
$$\boxed{\Psi_a(P;\mathcal{G}_2^*) = \sum_M \mathbb{E}[Y \mid A = a, M] \sum_O P(M \mid O, a)P(O).}$$

Examples (ii.c)



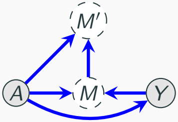
$\mathcal{G}_3$

Examples (ii.c)



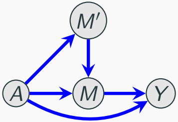
$\mathcal{G}_3$

$\overset{\mathrm{c}}{\sim}$



$\check{\mathcal{G}}_3$

Examples (ii.c)



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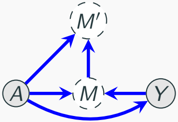
$\overset{c}{\sim}$



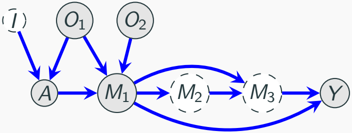
$\check{\mathcal{G}}_3$

$\Rightarrow$

$\mathcal{G}_3^*$

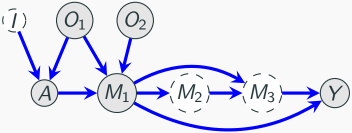
$$\Psi_a(P; \mathcal{G}_3^*) = \mathbb{E}[Y \mid A = a].$$

Examples (iii)



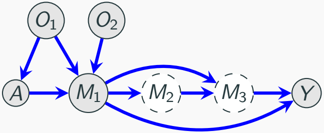
$\mathcal{G}$

Examples (iii)

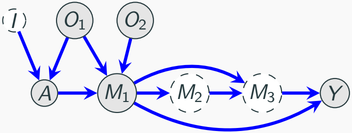


$\mathcal{G}$

$\Rightarrow$



Examples (iii)



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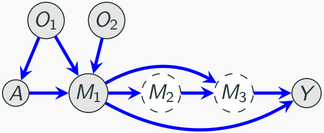
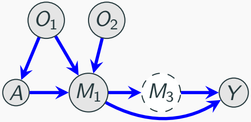


$\Rightarrow$



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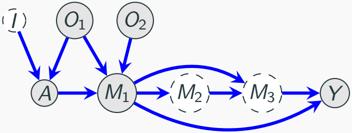

$\mathcal{G}$

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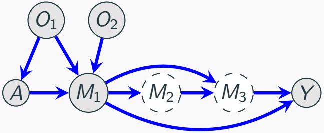
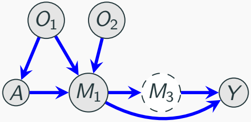
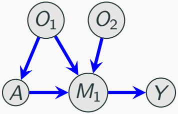


$\Rightarrow$



$\Rightarrow$



$\mathcal{G}^*$

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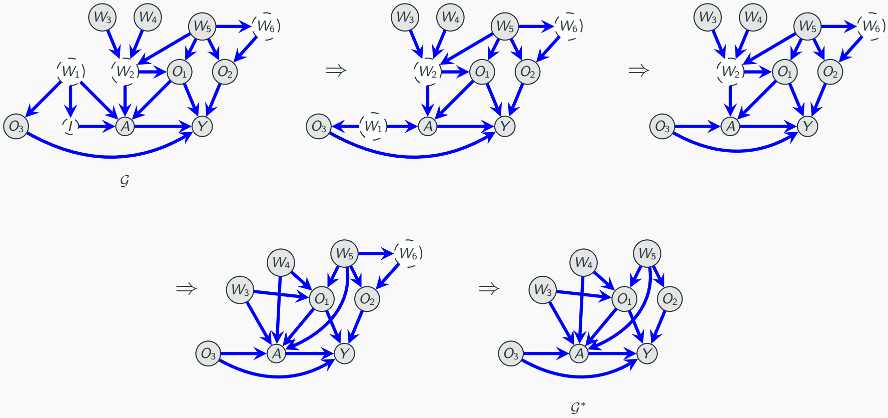
Examples (iv)


W_3 W_4 W_5 W_6
W_1 W_2 O_1 O_2
O_3 I A Y
$\mathcal{G}$
$\Rightarrow$
W_3 W_4 W_5 W_6
W_2 O_1 O_2
O_3 W_1 A Y
$\Rightarrow$
W_3 W_4 W_5 W_6
W_2 O_1 O_2
O_3 A Y
$\Rightarrow$
W_4 W_5 W_6
W_3 O_1 O_2
O_3 A Y
$\Rightarrow$
W_4 W_5
W_3 O_1 O_2
O_3 A Y
$\mathcal{G}^*$

reduceDAG

Try simplifying your causal DAG with R package **reduceDAG** available from https://unbiased.co.in

```
library(dagitty)
library(reduceDAG)
g <- dagitty('dag {
    A [pos="0,2", exposure]
    M [pos="1,1"]
    Y [pos="2,2", outcome]
    O [pos="1,0"]
    A -> M -> Y
    A -> Y
    O -> M
}')
cat(gFormula(g))
# sum_{M,Y} Y P[Y | A=a,M] sum_{O} P(M | A=a,O) P(O)
h <- reduceDAG(g, verbose=TRUE)
# Uninformative variables {M} are eliminated.
# Reduced g-formula:
# sum_{O,Y} Y P[Y | A=a,O] P(O)
```

Conclusion

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  ☞ $\mathcal{G}^*$ prescribes the simplest g-formula that is efficient.
  ☞ $\mathcal{G}^*$ could inform data collection and estimation strategies.

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  ☞ $\mathcal{G}^*$ could inform data collection and estimation strategies.
- R package `reduceDAG`.

Thanks!

arXiv: 2202.11994

R package: `reduceDAG`

References

- Andersson, Steen A., David Madigan, and Michael D. Perlman (1997). "A characterization of Markov equivalence classes for acyclic digraphs". In: *The Annals of Statistics* 25, pp. 505–541.
- Chickering, David Maxwell (1995). "A transformational characterization of equivalent Bayesian network structures". In: *Proceedings of UAI 1995*.
- Evans, Robin J. (2016). "Graphs for margins of Bayesian networks". In: *Scandinavian Journal of Statistics* 43.3, pp. 625–648.
- Guo, F. Richard and Emilija Perković (2021). "Minimal enumeration of all possible total effects in a Markov equivalence class". In: *Proceedings of the International Conference on Artificial Intelligence and Statistics*.

References ii

Henckel, Leonard, Emilija Perković, and Marloes H. Maathuis (2022). "Graphical criteria for efficient total effect estimation via adjustment in causal linear models". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* published online.

Pearl, Judea (1993). "Comment: graphical models, causality and intervention". In: *Statistical Science* 8.3, pp. 266–269.

— (1995). "Causal diagrams for empirical research". In: *Biometrika* 82.4, pp. 669–688.

— (2000). *Causality: Models, Reasoning, and Inference*. 1st. Cambridge University Press.

Robins, James M. (1986). "A new approach to causal inference in mortality studies with a sustained exposure period-application to control of the healthy worker survivor effect". In: *Mathematical Modelling* 7, pp. 1393–1512.

Rotnitzky, Andrea and Ezequiel Smucler (2020). "Efficient Adjustment Sets for Population Average Causal Treatment Effect Estimation in Graphical Models". In: *Journal of Machine Learning Research* 21.188, pp. 1–86.

Spirtes, Peter, Clark Glymour, and Richard Scheines (2000). *Causation, Prediction, and Search*. New York: Springer-Verlag.

Verma, Thomas and Judea Pearl (1991). *Equivalence and synthesis of causal models*. Tech. rep. R-150. Department of Computer Science, University of California, Los Angeles.

W-criterion Suppose $\{W_j\} \cup \mathrm{Ch}(W_j) \cap \mathbf{W}$ is topologically sorted as $\{W_{j_0} \equiv W_j, W_{j_1}, \ldots, W_{j_r}\}$. Then $W_j \in \mathbf{W} \setminus \mathbf{O}$ is uninformative if and only if

1. $W_j \perp\!\!\!\perp_{\mathcal{G}} \mathbf{O} \mid W_{j_r}, \mathrm{Pa}(W_{j_r}) \setminus \{W_j\}$,
2. and for $m = 1, \ldots, r$:
   - (i) $W_{j_{m-1}} \to W_{j_m}$ (children are chained)
   - (ii) $\mathrm{Pa}(W_{j_m}) \subseteq \mathrm{Pa}(W_{j_{m-1}}) \cup \{W_{j_{m-1}}\}$ (parent sets are decreasing)
   - (iii) $\mathrm{Pa}(W_{j_{m-1}}) \setminus \mathrm{Pa}(W_{j_m}) \perp\!\!\!\perp_{\mathcal{G}} \mathbf{O} \mid \mathrm{Pa}(W_{j_m})$ (left-over piece is separated from $\mathbf{O}$)

M-criterion Suppose $\{M_i\} \cup \mathrm{Ch}(M_i) \cap \mathbf{M}$ is topologically sorted as $\{M_{i_0} \equiv M_i, M_{i_1}, \ldots, M_{i_k}\}$. Then $M_i \in \mathbf{M}$ is uninformative if and only if

1. $M_i \perp\!\!\!\perp_{\mathcal{G}} \{A, Y\} \cup \mathbf{O}_{\min} \mid M_{i_k}, \mathrm{Pa}(M_{i_k}) \setminus \{M_i\}$,
2. and for $l = 1, \ldots, k$:
   - (i) $M_{i_{l-1}} \to M_{i_l}$ (children are chained)
   - (ii) $\mathrm{Pa}(M_{i_l}) \subseteq \mathrm{Pa}(M_{i_{l-1}}) \cup \{M_{i_{l-1}}\}$ (parent sets are decreasing)
   - (iii) $\mathrm{Pa}(M_{i_{l-1}}) \setminus \mathrm{Pa}(M_{i_l}) \perp\!\!\!\perp_{\mathcal{G}} \{A, Y\} \cup \mathbf{O}_{\min} \mid \mathrm{Pa}(M_{i_l})$ (left-over piece is separated from $A, Y, \mathbf{O}_{\min}$)

Nonparametric model $\mathcal{M}_0(\mathbf{V}) \equiv \{\text{all laws over vector } \mathbf{V}\}$.

Identifying formula Fix a model $\mathcal{M}(\mathbf{V}) \subseteq \mathcal{M}_0(\mathbf{V})$ and a functional $\gamma(P) : \mathcal{M}(\mathbf{V}) \to \mathbb{R}$.

Functional $\chi(P) : \mathcal{M}_0(\mathbf{V}) \to \mathbb{R}$ is an identifying formula for $\gamma(P)$ if $\chi(P) = \gamma(P)$ for every $P \in \mathcal{M}(\mathbf{V})$.

Efficient identifying formula Consider a semiparametric model $\mathcal{M}(\mathbf{V}) \subseteq \mathcal{M}_0(\mathbf{V})$ and a regular functional $\gamma : \mathcal{M}(\mathbf{V}) \to \mathbb{R}$. Let $\gamma^1_{P,\text{eff}}(\mathbf{V})$ be its efficient influence function with respect to $\mathcal{M}(\mathbf{V})$.

An identifying formula $\chi : \mathcal{M}_0(\mathbf{V}) \to \mathbb{R}$ for functional γ is called efficient if $\chi^1_{P,\text{NP}}(\mathbf{V}) = \gamma^1_{P,\text{eff}}(\mathbf{V})$ P-almost-everywhere for every $P \in \mathcal{M}(\mathbf{V})$.