BIOST 578: Special Topics
Causal inference in biomedical studies

Richard Guo

Lecture # 5: Causal DAGs and their variants

April 8, 2024


ZOO

Overview

A zoo of graphical models (non-causal or causal) and a myriad of acronyms:

- ADMG
- PAG
- MAG
- **DAG**
- chain graph
- CPDAG
- UG
- ancestral graph
- factor graph
- path diagram

...

Overview

For this quick intro, we shall focus on DAG and its variant ADMG (aka DAG with latents).

1. DAG as a probability model
2. DAG as a causal model
3. DAG as a tool for practitioners

DAG as a probability model

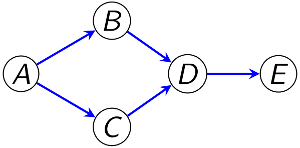
DAG

A graph $\mathcal{G}$ that consists of

- vertices $\boldsymbol{V}$,
- directed edges $\boldsymbol{E}$

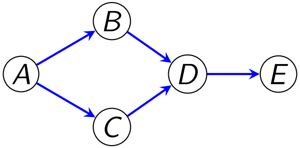
such that there is no directed cycle.

DAG



- $\mathrm{Pa}(D) = \{B, C\}$
- $\mathrm{Ch}(A) = \{B, C\}$
- $A \to B \to D \to E$ is a directed path $\quad A \in \mathrm{An}(E)$ and $E \in \mathrm{De}(A)$
- A and B are adjacent

DAG



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- Topological ordering: $A \prec B \prec C \prec D \prec E$ (not unique) such that

$$i \text{ and } j \text{ are adjacent with } i \prec j \implies i \to j.$$

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Then a DAG $\mathcal{G} = (\boldsymbol{V}, \boldsymbol{E})$ is associated with

$$\begin{aligned} \mathcal{M}_{\mathcal{G}} &:= \{P : p(\boldsymbol{V}) \text{ factorizes according to } \mathcal{G}\} \\ &= \left\{P : p(\boldsymbol{V}) = \prod_{v \in \boldsymbol{V}} p(v \mid \mathrm{Pa}(v))\right\}. \end{aligned}$$

▶ Bayesian network. ▶ semiparametric model

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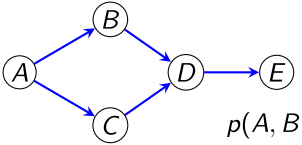
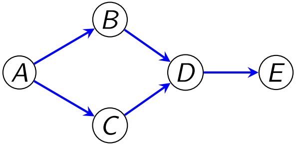


$$p(A, B, C, D, E) = p(A)\, p(B \mid A)\, p(C \mid A)\, p(D \mid B, C)\, p(E \mid D)$$

Equivalent description: NPSEM-IE



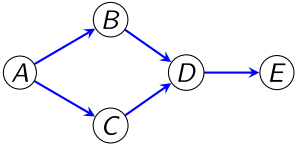
$$p(A, B, C, D, E) = p(A)\, p(B \mid A)\, p(C \mid A)\, p(D \mid B, C)\, p(E \mid D).$$

is equivalent to positing a nonparametric structural equation model with independent errors (NPSEM-IE):

$$\begin{aligned}
\varepsilon_a, \varepsilon_b, \varepsilon_c, \varepsilon_d, \varepsilon_e &\overset{\text{iid}}{\sim} \mathrm{unif}(0, 1) \\
A &= f_a(\varepsilon_a) \\
B &= f_b(A, \varepsilon_b) \\
C &= f_c(A, \varepsilon_c) \\
D &= f_d(B, C, \varepsilon_d) \\
E &= f_e(D, \varepsilon_e)
\end{aligned}$$

Constraints: missing edges

Topological ordering: $A \prec B \prec C \prec D \prec E$



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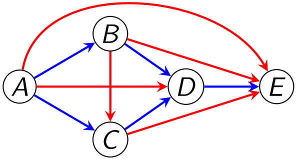

$$p(A)\, p(B \mid A)\, p(C \mid A, {\color{red}B})\, p(D \mid B, C, {\color{red}A})\, p(E \mid D, {\color{red}A, B, C})$$

- The full DAG represents any P
- the **nonparametric** model.

Conditional independence

A DAG $\mathcal{G}$, as a probability model $\mathcal{M}_{\mathcal{G}}$, posits

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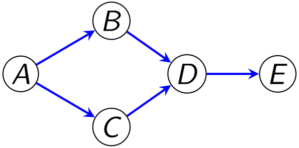
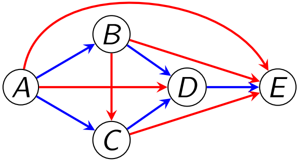


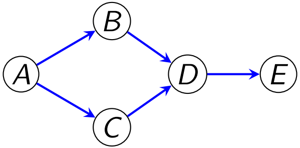


The missing '$B \to C$' posits

$$P(C \mid A, B) = P(C \mid A) \iff \boxed{B \perp\!\!\!\perp C \mid A} \iff P(B, C \mid A) = P(B \mid A)P(C \mid A).$$

Conditional independence

The graph



also implies, e.g.,

$$A, B, C \perp\!\!\!\perp E \mid D, \quad A, C \perp\!\!\!\perp E \mid B, D, \quad \ldots$$

☞ How we read off all the CIs a DAG implies ?

Dependence: mechanisms

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(2) Common cause (unconditionally)

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd} \qquad A \not\perp\!\!\!\perp B$$

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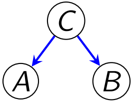
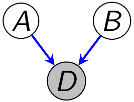


$A \not\perp\!\!\!\perp B$

(3) Conditioning on a common effect



$A \not\perp\!\!\!\perp B \mid D$

d-connecting path

- A **path** between A and B: a sequence of distinct, adjacent vertices

$$A \to \circ \to \circ \leftarrow \cdots \to B,$$

where every non-endpoint vertex is either a **collider** ($\to \circ \leftarrow$) or a **non-collider** ($\to \circ \to$, $\leftarrow \circ \leftarrow$, $\leftarrow \circ \to$)

A path is **d-connecting given $\boldsymbol{C}$** if

1. every non-collider $\notin \boldsymbol{C}$, and
2. every collider is $\in \boldsymbol{C}$ or is an ancestor of $\boldsymbol{C}$.

d-separation

Vertex A and vertex B are d-separated by vertex set $\boldsymbol{C}$, written as $A \perp\!\!\!\perp_{\mathcal{G}} B \mid \boldsymbol{C}$, if there is no d-connecting path between A and B given $\boldsymbol{C}$.

- Extended to $\boldsymbol{A} \perp\!\!\!\perp_{\mathcal{G}} \boldsymbol{B} \mid C$ for disjoint vertex sets $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$.

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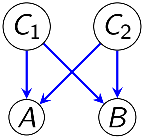

Global Markov property

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$$\boldsymbol{A} \perp\!\!\!\perp_{\mathcal{G}} \boldsymbol{B} \mid \boldsymbol{C} \implies \boldsymbol{A} \perp\!\!\!\perp \boldsymbol{B} \mid \boldsymbol{C} \; [P], \quad P \in \mathcal{M}_{\mathcal{G}}.$$

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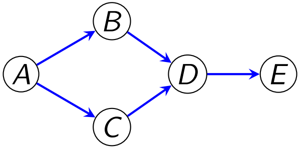

DAG as a CI model

- The global Markov property also holds **reversely**. If P satisfies

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Theorem Factorization $\iff$ Global Markov $\iff$ Local Markov.

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☞ That is, the model defined as $\mathcal{M}_{\mathcal{G}} := \{P : P \text{ factorizes according to } \mathcal{G}\}$ can be viewed as a **CI model**

$$\{P : \boldsymbol{A} \perp\!\!\!\perp_{\mathcal{G}} \boldsymbol{B} \mid \boldsymbol{C} \implies \boldsymbol{A} \perp\!\!\!\perp \boldsymbol{B} \mid \boldsymbol{C}\ [P]\},$$

i.e.,

$$\{P : P \text{ satisfies CIs that are encoded as d-separations in } \mathcal{G}\}.$$

Graphoid axioms

- From a set of CIs, new CIs may be derived, e.g., with applications of ‘graphoid axioms’:
  1. Symmetry: $A \perp\!\!\!\perp B \mid \boldsymbol{C} \implies B \perp\!\!\!\perp A \mid \boldsymbol{C}$
  2. Decomposition: $A \perp\!\!\!\perp B, D \mid \boldsymbol{C} \implies A \perp\!\!\!\perp B \mid \boldsymbol{C}$ and $A \perp\!\!\!\perp D \mid \boldsymbol{C}$
  3. Weak union: $A \perp\!\!\!\perp B, D \mid \boldsymbol{C} \implies A \perp\!\!\!\perp B \mid D, \boldsymbol{C}$
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☞ **Example** Given

$$A \perp\!\!\!\perp B, \quad A \perp\!\!\!\perp C \mid B,$$

we can derive

$$A \perp\!\!\!\perp B, \quad A \perp\!\!\!\perp C \mid B \implies A \perp\!\!\!\perp B, C \implies A \perp\!\!\!\perp C.$$

Completeness of d-separation

- **Question**: From the list CIs encoded by d-separations, can we derive a new CI (e.g., with graphoid axioms) that holds for **every** $P \in \mathcal{M}_{\mathcal{G}}$ but does not correspond to any d-separation in the graph? **NO!**

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Theorem For every $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ such that $\boldsymbol{A}$ and $\boldsymbol{B}$ are d-connected given $\boldsymbol{C}$ on $\mathcal{G}$, there exists $P \in \mathcal{M}_{\mathcal{G}}$ such that

$$\boldsymbol{A} \not\perp\!\!\!\perp \boldsymbol{B} \mid \boldsymbol{C} \; [P].$$

Completeness of d-separation

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Milan Studenỳ (1992) showed that CIs cannot be axiomatized by a finite set of rules. That is, one cannot deduce all the consequences of an arbitrary set $\{\mathrm{CI}_1, \mathrm{CI}_2, \ldots, \mathrm{CI}_k\}$ using a finite number of rules (e.g. graphoid axioms).
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☞ **Graphoid axioms are incomplete and cannot be completed**, if one is free to specify the list of CIs.

☞ However, DAG models are a class of nice CI models by confining the set of CIs (reducing complexity).

Examples over three variables

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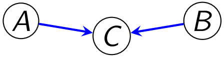
$$A \perp\!\!\!\perp B$$



Examples over three variables

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$A \perp\!\!\!\perp B$	$A \rightarrow C \leftarrow B$
$A \perp\!\!\!\perp B \mid C$	$A \rightarrow C \rightarrow B$ $A \leftarrow C \leftarrow B$ $A \leftarrow C \rightarrow B$

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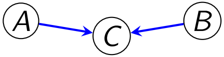

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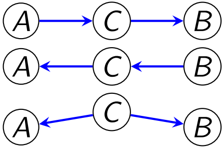

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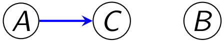

Markov equivalence

- $\mathcal{G}$ and $\mathcal{G}'$ are called '**Markov equivalent**', written as $\mathcal{G} \sim \mathcal{G}'$, if they define the same model.
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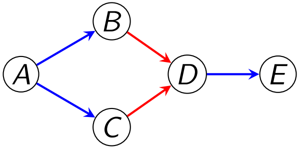
Theorem Two DAGs over the same set of vertices are **Markov equivalent** iff they share the same **adjacencies** and **unshielded colliders**.

- Unshielded collider: $B \to D \leftarrow C$ but B, C are not adjacent



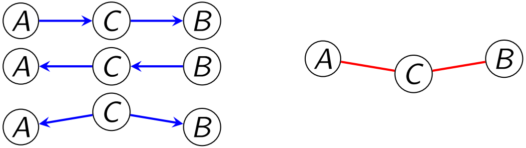
Markov equivalence class

- A Markov equivalence class can be represented by an essential graph / CPDAG.

(Without extra assumptions, DAGs can only be learned from data up to Markov equivalence.)

$A \perp\!\!\!\perp B$

$A \perp\!\!\!\perp B \mid C$



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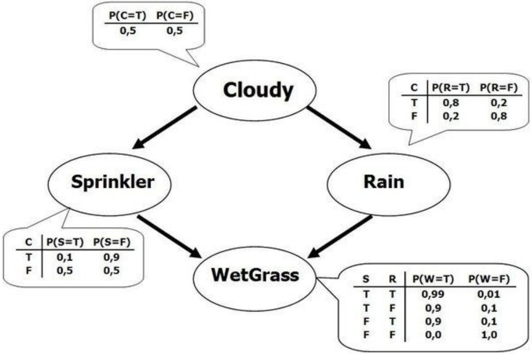
Parametric case: finite state space

- When every variable only takes finitely many levels, the model can be parametrized in terms of conditional probability tables $\{p(A \mid \mathrm{Pa}(A)) : A \in \boldsymbol{V}\}$.



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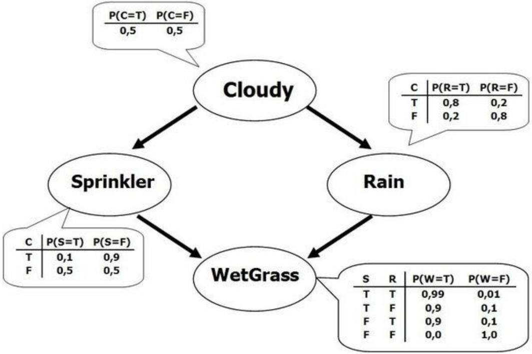


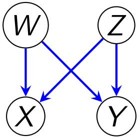
- Efficient algorithms exist for **marginalization** and computing **posterior probabilities**

Parametric case: linear SEM

Each linear equation posits that

$$V_i = \beta_i^{\mathsf{T}} \operatorname{Pa}(V_i) + \varepsilon_i,$$

where ε_i is exogenous error (drawn independently).



$$\begin{aligned} W &= && && \varepsilon_w \\ Z &= && && \varepsilon_z \\ X &= \beta_{wx} W && + \beta_{zx} Z + && \varepsilon_x \\ Y &= \beta_{wy} W && + \beta_{zy} Z + && \varepsilon_y \end{aligned}$$

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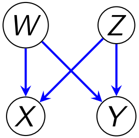

$$\begin{aligned} W &= & & & \varepsilon_w \\ Z &= & & & \varepsilon_z \\ X &= \beta_{wx} W & +\beta_{zx} Z + & & \varepsilon_x \\ Y &= \beta_{wy} W & +\beta_{zy} Z + & & \varepsilon_y \end{aligned}$$

- Because of acyclicity, it admits a unique solution:

$$\boldsymbol{V} = \boldsymbol{B}^\intercal \boldsymbol{V} + \boldsymbol{\varepsilon} \iff \boldsymbol{V} = (\boldsymbol{I} - \boldsymbol{B})^{-\intercal} \boldsymbol{\varepsilon}.$$

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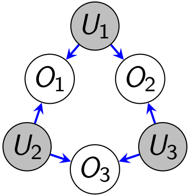
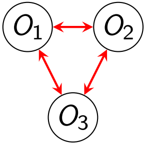
- $\boldsymbol{U}$ are **latent variables**





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\begin{aligned}
O_1 &\perp\!\!\!\perp O_2, \quad & O_1 &\not\!\perp\!\!\!\perp O_2 \mid O_3 \\
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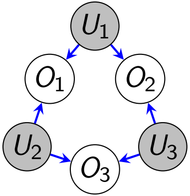
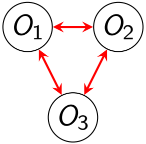
O_2 &\perp\!\!\!\perp O_3, \quad & O_2 &\not\!\perp\!\!\!\perp O_3 \mid O_1
\end{aligned}
$$

Limitation of DAGs

DAGs are not closed (in general) under marginalization and selection.

- Marginalization: Consider $\boldsymbol{V} = \boldsymbol{O} \cup \boldsymbol{U}$ for $\boldsymbol{O} \cap \boldsymbol{U} = \emptyset$, where we only get to observe $P(\boldsymbol{O})$.
  - $\boldsymbol{U}$ are **latent variables**





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☞ Such a CI model does not correspond to any DAG over $\boldsymbol{O}$.

DAGs with latent variables

For a DAG over $\boldsymbol{V} = \boldsymbol{O} \cup \boldsymbol{U}$,

$$\text{constraints in } P(\boldsymbol{O}) = \underbrace{\overbrace{\text{CIs}}^{\text{ancestral graphs}} + \text{`Verma' constraints}}_{\text{equalities (nested Markov models)}} + \text{inequalities}$$

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☞ See also Richardson and Spirtes (2002), Richardson (2003), Robin J Evans (2016), and Richardson, Robin J. Evans, et al. (2023).

DAG as a causal model

What makes it causal?

We have already seen that a DAG is a probability model as it defines a set of probability distributions $\mathcal{M}_{\mathcal{G}}$.

- $P \in \mathcal{M}_{\mathcal{G}}$ is an observed distribution over factual random variables.

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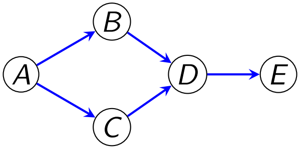
  3. connects the counterfactual distributions with the observed distribution.

Sampling from a DAG

$$\begin{tikzcd}[row sep=small] & B \arrow[dr] & & \\ A \arrow[ur] \arrow[dr] & & D \arrow[r] & E \\ & C \arrow[ur] & & \end{tikzcd}$$

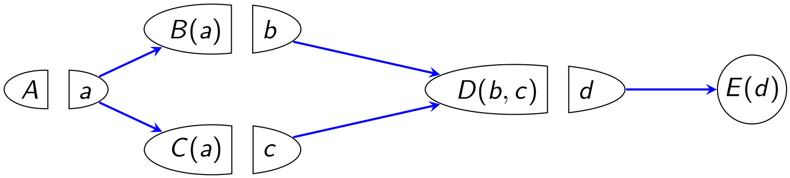
Sampling from a DAG



Following the topological ordering $A \prec B \prec C \prec D \prec E$,

1. Draw $A \sim P(A)$
2. Draw $B \sim P(B \mid A)$, $C \sim P(C \mid A)$
3. Draw $D \sim P(D \mid B, C)$
4. Draw $E \sim P(E \mid D)$

Alternative sampling (I): one-step-ahead counterfactuals



- Single-World Intervention Graph (SWIG) (Richardson and J. M. Robins, 2013)

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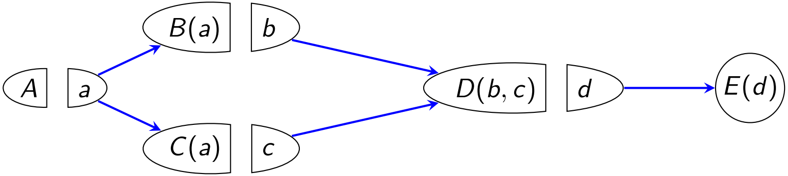

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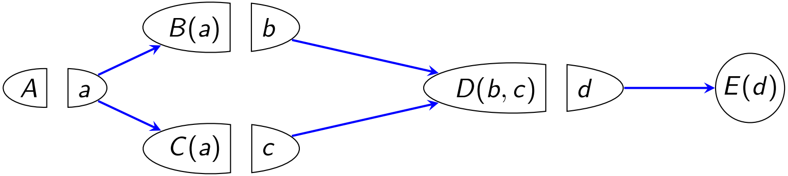

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1. Draw $A \sim P(A)$
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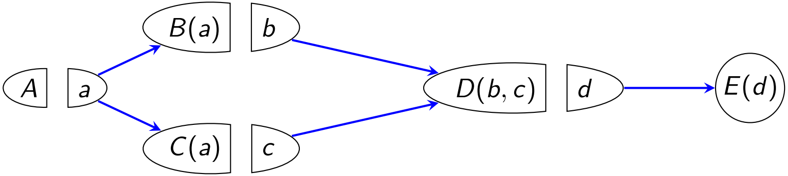

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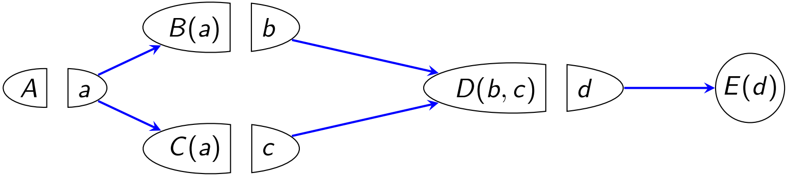

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Alternative sampling (I): one-step-ahead counterfactuals

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- This is called 'single-world' because we only posit that

$$A \perp\!\!\!\perp B(a), C(a) \quad \text{for every } a$$

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- Refrain from making 'cross-world' statements such as

$$A \perp\!\!\!\perp B(a), B(a'), B(a''), C(a), C(a'), C(a'')$$

because we will never see $B(a)$ and $B(a')$ together for $a \neq a'$.

☞ Cross-world assumptions cannot be empirically verified.

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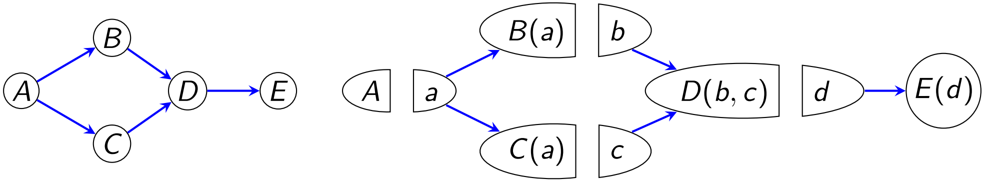
- Nevertheless, we can empirically examine $A \perp\!\!\!\perp B(a), C(a)$, if we can observe the naturally occurring value of A immediately before we intervene on it.

Alternative sampling (II): recursive substitution



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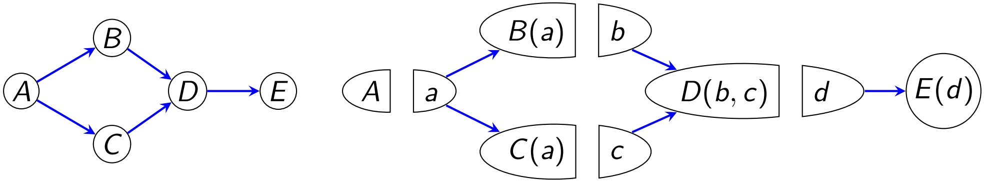

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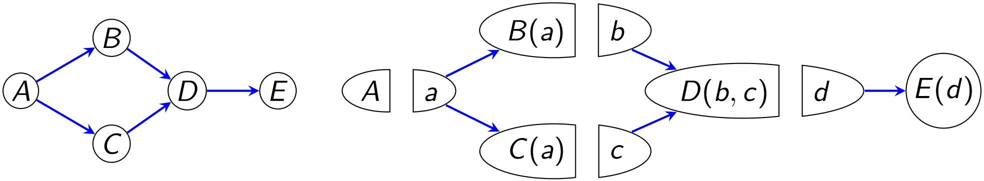

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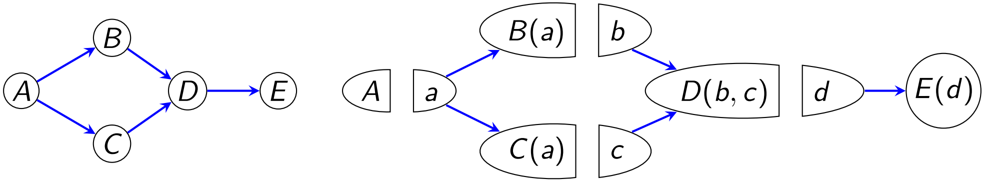

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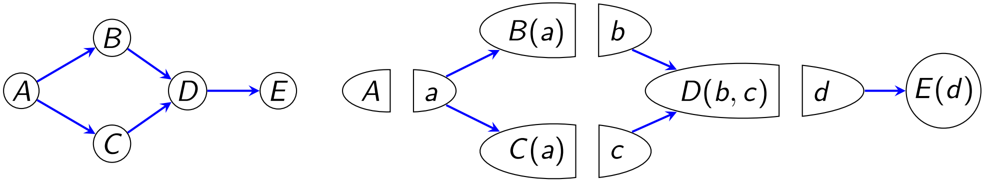

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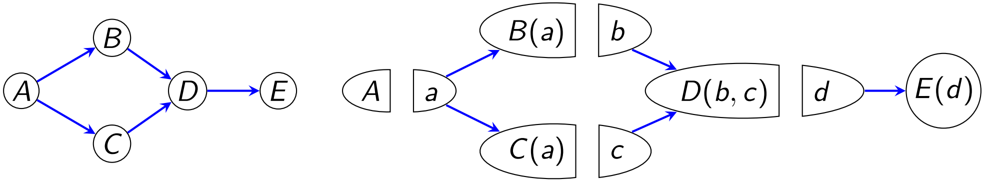

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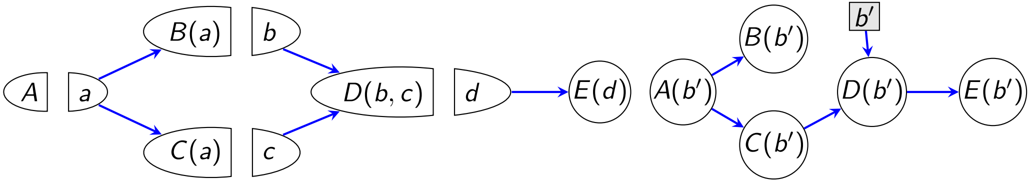
- Apparently, $(A, B, C, D, E) \sim P$

Alternative sampling (III): intervention

☞ Suppose that we **intervene on B and set it to b'** — imposes input to B's children.



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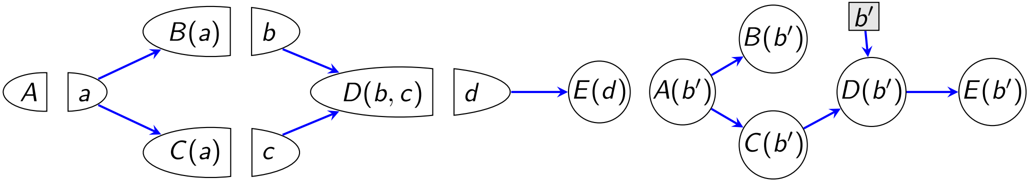

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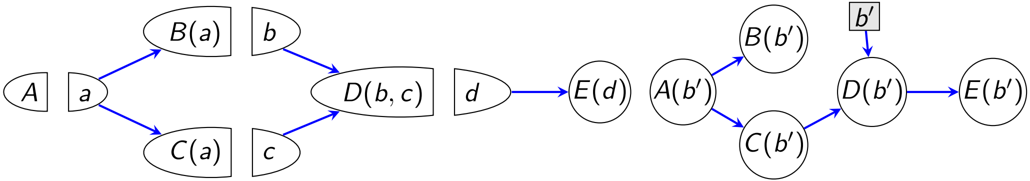

1. $A(b') = A$,
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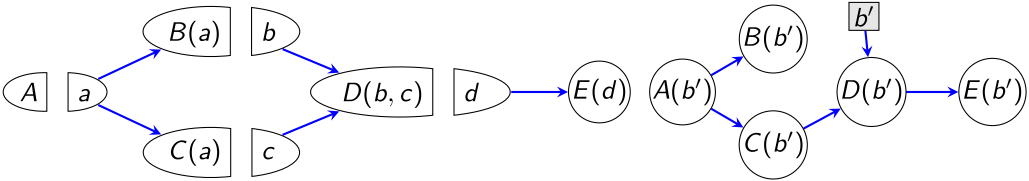

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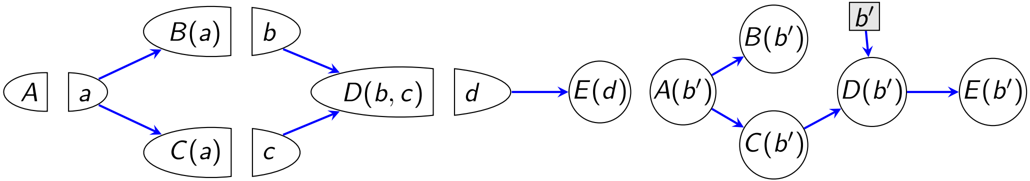

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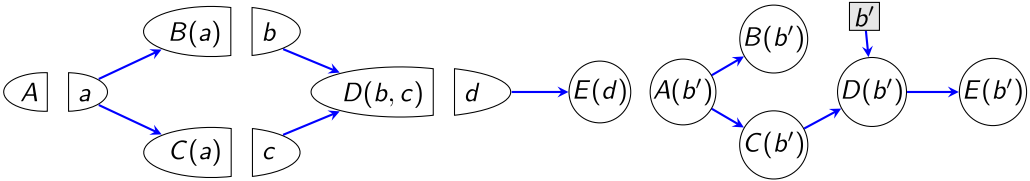

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- This defines the distribution of $P((A,B,C,D,E)(b'))$, or $P(A,B,C,D,E \mid \text{do}(B=b'))$.

Alternative sampling: the causal model

☞ This set of semantics defines the FFRCISTG / SWIG causal model associated with a DAG $\mathcal{G}$.

- 'Finest Fully Randomized Causally Interpreted Structured Tree Graph' (J. Robins, 1986)

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☞ It makes **weaker assumptions** than Pearl's NPSEM-IE (nonparametric structural equation model with independent errors) causal model.

- Even though NPSEM-IE and DAG define the same **probability model**!

g-formula

g-formula

With the semantics just described, the counterfactual distribution is

$$P\left(A(b') = a, B(b') = b, C(b') = c, D(b') = d, E(b') = e\right) =$$
$$P(A = a)\, P(B = b \mid A = a)\, P(C = c \mid A = a)\, P(D = d \mid B = b', C = c)\, P(E = e \mid D = e).$$

- This is **identified** from the observed distribution P because every **one-step-ahead conditional** is identified from P.

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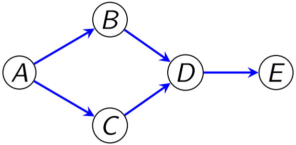
g-formula

$$P(\boldsymbol{V}(a) = \boldsymbol{v}) = \prod_{i=1}^{|V|} P(v_i \mid a_{\mathrm{Pa}(i)\cap A}, v_{\mathrm{Pa}(i)\setminus A})$$

- From this we can identify counterfactual means, etc.

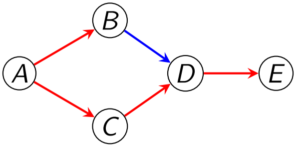
g-formula



- Example

$$
\begin{aligned}
\mathbb{E}\, E(b') &= \sum_{a,b,c,d,e} eP(A=a)\, P(B=b \mid A=a)\, P(C=c \mid A=a) \\
&\qquad \times P(D=d \mid B=b', C=c)\, P(E=e \mid D=d) \\
&= \sum_{d,c} \mathbb{E}[E \mid D=d] P(D=d \mid B=b', C=c) P(C=c) \\
&= \sum_{d,c} \mathbb{E}[E \mid D=d, B=b', C=c] P(D=d \mid B=b', C=c) P(C=c) \quad (why?)
\end{aligned}
$$

g-formula



$$
\begin{aligned}
\mathbb{E}\, E(b') &= \sum_c \sum_d \mathbb{E}[E \mid D = d, B = b', C = c] P(D = d \mid B = b', C = c) P(C = c) \\
&= \sum_c \mathbb{E}[E \mid B = b', C = c] P(C = c) \\
&= \mathbb{E}\left\{\mathbb{E}[E \mid B = b', C = c]\right\}.
\end{aligned}
$$

- Backdoor/adjustment formula that adjusts for C

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- A set of variables $\boldsymbol{S} \subseteq \boldsymbol{V} \setminus \{A, Y\}$ is a valid adjustment set if

$$P(Y(a) = y) = \mathbb{E}\{P(Y = y \mid A = a, \boldsymbol{S})\}$$

under the SWIG causal model associated with $\mathcal{G}$.

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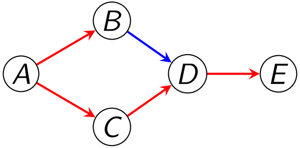
under the SWIG causal model associated with $\mathcal{G}$.

Theorem $\boldsymbol{S}$ is a valid adjustment set if it satisfies the backdoor criterion:

1. $\boldsymbol{S}$ contains no descendant of A,
2. No d-connecting path between A and Y given $\boldsymbol{S}$ with an arrowhead into A.

Backdoor/adjustment

- Identifying $P(E(b'))$.



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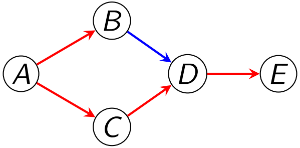

- Valid adjustment sets:

$$\{C\}, \quad \{A\}, \quad \{A, C\}.$$

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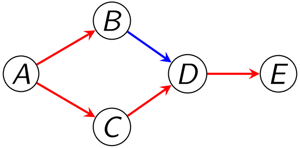

- Valid adjustment sets:

$$\{C\}, \quad \{A\}, \quad \{A, C\}.$$

☞ Adjusting for $\{C\}$ is the most efficient (Henckel et al., 2022; Rotnitzky and Smucler, 2020).

- A different, but more efficient estimator uses C, D (Guo, Perković, et al., 2023).

In the presence of latent variables

- For a DAG with latent variables $\boldsymbol{U}$, we can use latent projection to obtain an ADMG (acyclic directed mixed graph) over observed variables.

1. Whenever there is a path of the form $w \to u_1 \to \cdots \to u_2 \to v$ add $w \to v$ (if not already present).
2. Whenever there is a path of the form $w \leftarrow u_1 \leftarrow \cdots \to u_2 \to v$ add $w \leftrightarrow v$.

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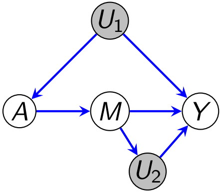
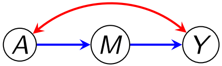
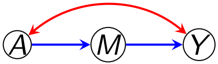
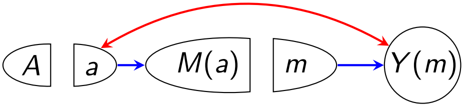
- Example





Front-door formula

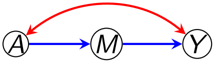


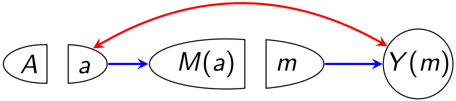


Front-door formula

$$\mathbb{E}\,Y(a) = \sum_m \overbrace{\mathbb{E}\{\mathbb{E}[Y \mid M = m, A]\}}^{\mathbb{E}\,Y(m)} \times P(M = m \mid A = a).$$

Front-door formula





$$
\begin{aligned}
P(Y(a)=y) &= P(Y(M(a))=y) \\
&= \sum_m P(Y(M(a))=y \mid M(a)=m)\, P(M(a)=m) \quad \text{(why?)} \\
&= \sum_m P(Y(m)=y \mid M(a)=m)\, P(M(a)=m) \\
&= \sum_m P(Y(m)=y)\, P(M(a)=m) \quad \text{(why?)} \\
&= \sum_m \left\{ \sum_{a'} P(Y=y \mid M=m, A=a') P(A=a') \right\} P(M(a)=m \mid A=a) \quad \text{(why?)} \\
&= \sum_m \left\{ \sum_{a'} P(Y=y \mid M=m, A=a') P(A=a') \right\} P(M=m \mid A=a).
\end{aligned}
$$

Identification in the presence of latent variables

- Not all (single-world) counterfactual quantities are identified.
- The ID algorithm due to Jin Tian provides a **complete solution**.
  - See also Shpitser and Pearl (2006), Richardson, Robin J. Evans, et al. (2023, §4.3).

DAG as a tool for practitioners

Use of DAGs in practice

Practitioners can use DAGs/ADMGs to

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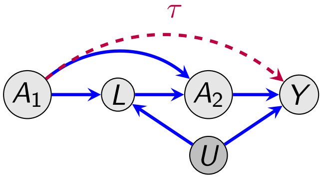

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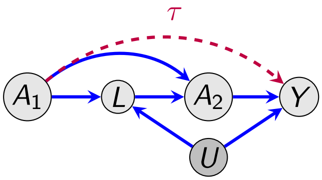

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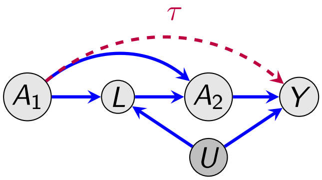

2. **Identify** a causal quantity of interest and consider its **estimation**. (to some extent)
3. **Design** an observational study. (largely open)

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4. Drawing a DAG can be a bad idea: unreliable and unnecessary.
5. Model elicitation, robust methods, sensitivity analysis.
    - See Guo and Zhao (2023) for an interactive protocol of eliciting an adjustment set.

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