Richard Guo's contribution to the Discussion of 'Parameterizing and Simulating from Causal Models' by Evans and Didelez

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Oct 19, 2023

I congratulate Evans and Didelez warmly on this innovative and inspiring paper. Here, I offer my interpretation of the approach in terms of two factorizations and a density ratio.

Any distribution over Z, X, Y can be factorized in two ways:

$$\left(P_{ZX}, P_{Y|Z,X}\right) \underset{C}{\overset{C^{-1}}{\leftrightarrows}} P_{ZXY} \underset{A}{\overset{A^{-1}}{\rightleftarrows}} \left(P_{ZX},\, P_{Y|X},\, \phi_{ZY|X}\right). \tag{1}$$

Factorization C^{-1} is the usual one; it is factorization A^{-1} that is the focus of the two authors. More concretely, suppose $\phi_{ZY|X}(z, y \mid x)$ is a conditional copula density, i.e., for every value of x, $(z, y) \mapsto \phi_{ZY|X}(z, y \mid x)$ is a density function over $[0, 1]^2$ with uniform margins. To obtain $A^{-1}(P_{ZXY})$, take P_{ZX} and $P_{Y|X}$ to be the corresponding component and let $\phi_{ZY|X=x}$ be the density function of $(F(Z \mid X = x), F(Y \mid X = x))$ for every x in the support of X. Conversely, to compose P_{XYZ} from the three pieces, we have

$$\begin{aligned} p_{ZXY}(z, x, y) &= A\left(p_{ZX}, p_{Y|X}, \phi_{ZY|X}\right) \\ &= p_{ZX}(z, x)\, \underbrace{p_{Y|X}(y \mid x)\, \phi_{ZY|X}(F(z \mid x), F(y \mid x) \mid x)}_{=p_{Y|ZX}(y|z,x)}, \end{aligned} \tag{2}$$

where $F(z \mid x)$ and $F(y \mid x)$ are defined by P_{ZX} and $P_{Y|X}$ respectively. Note that although P_{ZX} appear on both sides of Eq. (1), the relation between $P_{Y|Z,X}$ and the pair $(P_{Y|X}, \phi_{ZY|X})$ is not a separate bijection because $F(z \mid x)$ is needed to map one to the other. In other words, the map between $P_{Y|Z,X}$ and $(P_{Y|X}, \phi_{ZY|X})$ itself depends on P_{ZX}.

Suppose P^*_{ZXY} is a related distribution, of which the margin $P^*_{Y|X}$ is our model of interest. We require that P^*_{ZXY} is related to P_{ZXY} through a density ratio r, given by

$$\frac{p^*(z, x, y)}{p(z, x, y)} = r(z, x; p),$$

such that (i) r does not depend on y, (ii) $r > 0$ P-almost everywhere, and (iii) r can be identified from P. Then, by integrating out y on both sides of $p^*(z, x, y) = r(z, x; p)\, p(z, x, y)$, we have

$$r(z, x; p) = \frac{p^*(z, x)}{p(z, x)}, \quad p^*(y \mid z, x) = p(y \mid z, x). \tag{3}$$

That $p^*(z, x, y)$ being "cognate" with respect to $p(z, x, y)$ amounts to choosing

$$r(z, x; p) = \frac{w(z \mid x)}{p(z \mid x)}$$

for some weight $w(z \mid x)$, such as $w(z \mid x) = p(z)$ for estimating ATE and $w(z \mid x) = p(z \mid x = 1)$ for estimating ATT.

With $r(x, y; p)$ chosen and fixed, the frugal parametrization is to represent p (and hence p^*) through p_{ZX}, $p^*_{Y|X}$ and $\phi^*_{ZY|X}$, i.e., the following three pieces in box:

$$\begin{aligned} \left(P_{ZX}, P_{Y|Z,X}\right) \overset{C^{-1}}{\underset{C}{\leftrightarrows}} P_{ZXY} \overset{A^{-1}}{\underset{A}{\rightleftarrows}} \left(\boxed{P_{ZX}}, P_{Y|X}, \phi_{ZY|X}\right) \\ \left(P^*_{ZX}, P^*_{Y|Z,X}\right) \overset{C^{-1}}{\underset{C}{\leftrightarrows}} P^*_{ZXY} \overset{A^{-1}}{\underset{A}{\rightleftarrows}} \left(P^*_{ZX}, \boxed{P^*_{Y|X}}, \boxed{\phi^*_{ZY|X}}\right). \end{aligned}$$

The likelihood can be obtained through

$$\begin{aligned} p(z, x, y) = \frac{p^*(z, x, y)}{r(z, x; p)} &= \frac{A\left(p^*(z, x), p^*(y \mid x), \phi^*(z, y \mid x)\right)}{r(z, x; p)} \\ &= \frac{A\left(p(z, x)\, r(z, x; p), p^*(y \mid x), \phi^*(z, y \mid x)\right)}{r(z, x; p)}, \end{aligned}$$

where the second line uses Eq. (3). When $\phi^*(z, y \mid x)$ is a conditional copula density, using Eq. (2), it follows that

$$p(z, x, y) = p(z, x) \underbrace{p^*(y \mid x)\, \phi^*(F^*(z \mid x), F^*(y \mid x) \mid x)}_{= p(y \mid z, x)}. \tag{4}$$

Indeed, by uniform margins of the copula, one can check that

$$\int p^*(y \mid x)\, \phi^*(F^*(z \mid x), F^*(y \mid x) \mid x)\, \mathrm{d}y = \int \mathrm{d}F^*(y \mid x)\, \phi^*(F^*(z \mid x), F^*(y \mid x) \mid x) = 1.$$

Further, in Eq. (4), the arguments of ϕ^* depends on $F^*(y \mid x)$ and $F^*(z \mid x)$: the former is derived from $p^*(y \mid x)$ and the latter is the conditional CDF pertaining to $p^*(z, x) = p(z, x)\, r(z, x; p)$, given by

$$F^*(z \mid x) = \frac{\int_{-\infty}^{z} p(z', x) r(z', x; p)\, \mathrm{d}z'}{\int_{-\infty}^{+\infty} p(z', x) r(z', x; p)\, \mathrm{d}z'}. \tag{5}$$

Hence, Eq. (4) provides an explicit expression for $p(z, x, y)$ in terms of p_{ZX}, $p^*_{Y|X}$ and $\phi^*_{ZY|X}$, which depends on the pre-specified density ratio $r(x, y; p)$ through Eq. (5). Multiplying Eq. (4) by the density ratio simply yields the expression for $p^*(z, x, y)$.

Example 1 (Sequentially randomized trial)**.** For Fig. 1, with $r(a, l, b; p) = p(b)/p(b \mid a, l)$, we can parametrize P and P^* in terms of the three pieces in box below:

$$\begin{aligned} P(A, L, B, Y) \overset{A^{-1}}{\underset{A}{\rightleftarrows}} \left(\boxed{P_{ALB}}, P_{Y|AB}, \phi_{YL|AB}\right) \\ P^*(A, L, B, Y) \overset{A^{-1}}{\underset{A}{\rightleftarrows}} \left(P^*_{ALB}, \boxed{P^*_{Y|AB}}, \boxed{\phi^*_{YL|AB}}\right). \end{aligned}$$

Example 2 (Partially marginal model)**.** Suppose we have an observational study with baseline covariates $Z = (Z_1, Z_2)$, treatment X and outcome Y. Imagine that we want to study how Z_1 modifies the effect of X on Y. Hence, we want to choose $P^*(Z, X, Y)$ such that $P^*(Y \mid X, Z_1)$ aligns with our intended marginal model $P(Y \mid Z_1, \mathrm{do}(X))$. In the meantime, we need to use

(a) (b)

Figure 1: (a) $P(A, L, B, Y)$, (b) $P^*(A, L, B, Y)$.

both Z_1 and Z_2 to control for the confounding between X and Y. This is called a "partially" marginal model because the marginal model is conditional on a partial collection of baseline covariates. To facilitate this analysis, we can choose density ratio

$$r(z_1, z_2, x; p) = p(x)/p(x \mid z_1, z_2)$$

and parametrize p (and hence p^*) in terms of

$$p(z_1, z_2, x), \quad p^*(y \mid z_1, x), \quad \phi^*_{Z_2,Y|Z_1,X}.$$